

EXTENSIONS OF HILBERT BIMODULES AND ASSOCIATED CUNTZ-PIMSNER ALGEBRAS

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ABSTRACT. We extend the definition of an extension of a right Hilbert module to the setting of Hilbert bimodules and show that an extension of Hilbert bimodules induces an extension of Cuntz-Pimsner algebras. We also study the Cuntz-Pimsner algebra associated to the multiplier bimodule and show that an extension can be realised as a restricted direct-sum bimodule.

1. INTRODUCTION

A C^* -correspondence is a pair (X, A) where X is a right Hilbert A -module with a left action of A on X . In [11], Pimsner first showed how to associate a C^* -algebra to certain C^* -correspondences. These C^* -algebras are now known as Cuntz-Pimsner algebras. Since then, this definition has been extended to all C^* -correspondences by Katsura in his series of papers [8, 9, 10].

When a C^* -correspondence is simultaneously a left Hilbert module, with a left inner-product, we call it a Hilbert bimodule. In this paper, we extend the definition of an extension of right Hilbert modules due to Bakić and Guljaš [2, 3], to the setting of Hilbert bimodules. We then show that an extension of Hilbert bimodules induces an extension of Cuntz-Pimsner algebras in a functorial way.

We begin Section 2 with the definition of Hilbert bimodules, due to Brown, Mingo and Shen [4]. We also recall the definition of the multiplier bimodule $M(X)$, first defined for imprimitivity bimodules in [7], which is a generalisation of the multiplier algebra for C^* -algebras.

In Section 3 we define extensions of Hilbert bimodules. This definition is more or less unchanged from the analogous definition for right Hilbert modules given in [3], we need only show that this definition also works for bimodules. As an example, we show that the multiplier bimodule $(M(X), M(A))$ is the largest essential extension of a full bimodule (X, A) . We also show that under certain assumptions, given an ideal $I \triangleleft A$ we can find a bimodule (X_I, I) with (X, A) as an extension. Furthermore, this extension will be essential if and only if I is an essential ideal of A . We then show that the classification of extensions of right Hilbert modules up to a Busby morphism from [3] extends to the category of Hilbert bimodules.

We continue in Section 4 with the definition of the C^* -algebra associated to a Hilbert bimodule. We show that this process is functorial, so morphisms between Hilbert bimodules induce C^* -algebra homomorphisms between Cuntz-Pimsner algebras. Furthermore, we show that extensions of Hilbert bimodules induce extensions of the associated Cuntz-Pimsner algebras. We also briefly study the Cuntz-Pimsner algebra $\mathcal{O}_{M(X)}$ associated to the multiplier bimodule, and show with an example that in general this is smaller than the multiplier algebra $M(\mathcal{O}_X)$.

2. PRELIMINARIES

For a C^* -algebra A , a *right Hilbert A -module* is a Banach space X equipped with a non-degenerate right action of A , and an A -valued inner-product $\langle \cdot, \cdot \rangle_X$ satisfying

- (1) $\langle \xi, \eta a \rangle_X = \langle \xi, \eta \rangle_X a$;
- (2) $\langle \eta, \xi \rangle_X = \langle \xi, \eta \rangle_X^*$; and
- (3) $\langle \xi, \xi \rangle_X \geq 0$ and $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle_X\|}$

for all $\xi, \eta \in X$ and $a \in A$.

Likewise, a *left Hilbert A -module* is a Banach space X with a non-degenerate left action of A and an A -valued inner product ${}_X\langle \cdot, \cdot \rangle$ satisfying analogous relations to those above; i.e.

- (1) ${}_X\langle a\xi, \eta \rangle = a_X \langle \xi, \eta \rangle$;
- (2) ${}_X\langle \eta, \xi \rangle = {}_X\langle \xi, \eta \rangle^*$; and
- (3) ${}_X\langle \xi, \xi \rangle \geq 0$ and $\|\xi\| = \sqrt{\|{}_X\langle \xi, \xi \rangle\|}$

for all $\xi, \eta \in X$ and $a \in A$.

The following definition of a Hilbert bimodule is originally due to Brown, Mingo and Shen [4].

Definition 2.1. We say that the pair (X, A) is a *Hilbert bimodule*, when X is simultaneously a left and right Hilbert A -module, and satisfies the relation

$${}_X\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_X.$$

We say a bimodule (X, A) is *full* if both ${}_X\langle X, X \rangle \subset A$ and $\langle X, X \rangle_X \subset A$ are dense. In the literature, a full bimodule is also known as an imprimitivity bimodule.

Given a C^* -algebra A and Hilbert bimodules (X, A) and (Y, A) , we denote by $\mathcal{L}(X, Y)$ the set of all *adjointable* operators from X to Y ; that is, linear operators $T : X \rightarrow Y$ such that there exists a linear operator $T^* : Y \rightarrow X$ called the *adjoint* of T satisfying

$$\langle T\xi, \eta \rangle_Y = \langle \xi, T^*\eta \rangle_X$$

for all $\xi \in X, \eta \in Y$. If the adjoint T^* exists, it is unique. We will write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. With the usual operator norm $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$, $\mathcal{L}(X)$ is a C^* -algebra

For $\xi \in X, \eta \in Y$, define $\theta_{\eta, \xi} \in \mathcal{L}(X, Y)$ to be the operator satisfying

$$\theta_{\eta, \xi}(\zeta) = \eta \langle \xi, \zeta \rangle_X.$$

This is an adjointable operator with $(\theta_{\eta, \xi})^* = \theta_{\xi, \eta}$. We call

$$\mathcal{K}(X, Y) = \overline{\text{span}}\{\theta_{\eta, \xi} : \xi \in X, \eta \in Y\}$$

the set of *compact* operators. It is easily seen that $\mathcal{K}(X, X) = \mathcal{K}(X)$ is a closed two-sided ideal in $\mathcal{L}(X)$.

The following simple example will be useful when we are defining the C^* -algebras associated to Hilbert bimodules.

Example 2.2. Let D be a C^* -algebra. Then the pair (D, D) is a Hilbert bimodule with left and right actions by given by multiplication and left and right inner products given by

$${}_D \langle a, b \rangle = ab^* \quad \text{and} \quad \langle a, b \rangle_D = a^*b$$

for $a, b \in D$. It is well-known that there are isomorphisms $\mathcal{K}(D) \cong D$ and $\mathcal{L}(D) = M(D)$.

Definition 2.3. Let (X, A) and (Y, B) be Hilbert bimodules. A morphism from (X, A) to (Y, B) is a pair of maps (ψ_X, ψ_A) where the map $\psi_X : X \rightarrow Y$ is linear and $\psi_A : A \rightarrow B$ is a C^* -homomorphism such that, for any $\xi, \eta \in X, a \in A$ we have

- (1) $\psi_X(\xi a) = \psi_X(\xi) \psi_A(a)$
- (2) $\psi_X(a \xi) = \psi_A(a) \psi_X(\xi)$
- (3) $\psi_A(\langle \xi, \eta \rangle_X) = \langle \psi_X(\xi), \psi_X(\eta) \rangle_Y$, and
- (4) $\psi_A({}_X \langle \xi, \eta \rangle) = {}_Y \langle \psi_X(\xi), \psi_X(\eta) \rangle$.

We say a morphism is *injective* if the map $\psi_A : A \rightarrow B$ is injective. In this case, ψ_X is also necessarily injective. For full Hilbert bimodules, [1, Theorem 2.3] implies that the converse is also true - that is; ψ_X is injective if and only if ψ_A is injective.

Definition 2.4. Let (X, A) be a Hilbert bimodule. Define the multiplier $M(X) := \mathcal{L}(A, X)$

The following proposition is already well-known, see [6] for example. A proof is given here for completeness.

Proposition 2.5. *Let (X, A) be a full Hilbert bimodule. Then there exist left and right actions, and left and right inner products such that the pair $(M(X), M(A))$ is also a Hilbert bimodule.*

Proof. Firstly, by identifying the multiplier algebra $M(A)$ with $\mathcal{L}(A)$ as in Example 2.2, we can define the right action of $M(A)$ on $M(X)$ simply as composition of operators.

For the left action, define a map $\phi : A \rightarrow \mathcal{L}(X)$ by

$$\phi(a)(\xi) = a\xi.$$

Since we have assumed that (X, A) is a full Hilbert bimodule, ϕ is in fact an isomorphism $A \rightarrow \mathcal{K}(X)$ and hence extends to an isomorphism $\bar{\phi} : M(A) \rightarrow \mathcal{L}(X)$. So define the left action by

$$(mT)(a) = \bar{\phi}(m)(T(a))$$

for $m \in M(A)$, $T \in M(X)$ and $a \in A$.

Define the right inner-product $\langle \cdot, \cdot \rangle_{M(X)}$ by

$$\langle S, T \rangle_{M(X)} = S^*T.$$

For the left inner-product, notice that for any $S, T \in M(X)$ we have $ST^* \in \mathcal{L}(X)$. So we may define the left action

$${}_{M(X)}\langle S, T \rangle := \bar{\phi}^{-1}(ST^*).$$

We need to see that this satisfies Definition 2.1. It has already been proven in [3] that the right module structure satisfies the appropriate conditions, so we instead concentrate on the left module structure.

Fix $m \in M(A)$ and $S, T \in M(X)$. Then

$$\begin{aligned} {}_{M(X)}\langle mS, T \rangle &= \bar{\phi}^{-1}((mS)T^*) \\ &= \bar{\phi}^{-1}(\bar{\phi}(m)ST^*) \\ &= m\bar{\phi}^{-1}(ST^*) \\ &= m\bar{\pi}(ST^*). \end{aligned}$$

Likewise, for R, S and T in $M(X)$ we have

$$\begin{aligned} {}_{M(X)}\langle R, S \rangle T &= \bar{\phi}(\bar{\pi}(RS^*))T \\ &= (RS^*) \circ T \\ &= R \circ (S^*T) \\ &= R\langle S, T \rangle_{M(X)} \end{aligned}$$

as required. □

3. EXTENSIONS OF HILBERT BIMODULES

We begin this section with the definition of an extension of a Hilbert bimodule. This is modelled on the definition given for right Hilbert modules in [3].

First we need the following concept. For a Hilbert bimodule (X, A) , we say an ideal I of A is *invariant with respect to X* if $IX = XI$.

Definition 3.1. Let (X, A) be a Hilbert bimodule. We say that a Hilbert bimodule (Y, B) is an *extension* of (X, A) if B is a C^* -algebra containing A as an ideal, and there is a morphism of Hilbert bimodules $(\psi_X, \psi_A) : (X, A) \rightarrow (Y, B)$ such that ψ_A is simply the inclusion of A in B and A is invariant with respect to Y . We will call (Y, B) an *essential extension* if A is an essential ideal in B .

Proposition 3.2. *Let (X, A) be a Hilbert bimodule, and let $I \triangleleft A$ be invariant with respect to X . Define*

$$X_I := \overline{\text{span}}\{\xi a : \xi \in X, a \in I\} \subset X.$$

Then (X_I, I) is a Hilbert bimodule, with left and right inner-products and left and right actions inherited from (X, A) . Furthermore, (X, A) is an extension of (X_I, I) , and is essential if and only if I is an essential ideal in A .

Proof. We must show that the both left and right actions have range in X_I and both left and right inner products have range in I . This is clear for the right action, and for the right inner product this follows easily from A -linearity. The left action follows precisely from the assumption that I is invariant with respect to X . Finally, for the left inner product, fix $\xi, \eta \in X_I$. Then the invariance of I with respect to X implies that there exists some $i \in I$ and $\zeta \in X$ satisfying $\xi = i\zeta$. So

$${}_X\langle \xi, \eta \rangle = {}_X\langle i\zeta, \eta \rangle = i_X\langle \zeta, \eta \rangle \in I$$

as required.

Lastly, given the definition of the inner products and actions, it is clear that the natural inclusion of (X_I, I) inside (X, A) is a morphism of Hilbert bimodules. \square

As an easy consequence of this proposition, we get the following.

Corollary 3.3. *Let (X, A) be a full Hilbert bimodule. Then the multiplier bimodule $(M(X), M(A))$ is an essential extension of (X, A) .*

Proof. In light of the previous proposition and the definition of the bimodule $(M(X), M(A))$, it is enough to show that A is an $M(X)$ -invariant ideal of $M(A)$. In fact we show that $M(X)\iota_A(A) = \iota_X(X) = \iota_A(A)M(X)$. Fix $a \in A$ and $T \in M(X)$. For any $b \in A$ we have

$$(T\iota_A(a))(b) = T(ab) = T(a)b = \iota_X(T(a))(b)$$

so $T\iota_A(a) = \iota_X(T(a)) \in \iota_X(X)$. Since X is full, the left action of A on X induces an isomorphism $A \rightarrow K(X)$, so a simple calculation shows that

$$\iota_A(A)M(X) = \mathcal{K}(X)\mathcal{L}(A, X) \subset \mathcal{K}(A, X)$$

Then from [12, Lemma 2.32] we have an isomorphism $\mathcal{K}(A, X) \rightarrow \iota_X(X)$, so we have shown that we have inclusions $\iota_A(A)M(X) \subset \iota_X(X)$ and $M(X)\iota_A(A) \subset \iota_X(X)$. For surjectivity, fix an approximate unit $\{e_i\}$ for A . Then it is straightforward to show that for any $\xi \in X$

$$\lim(\iota_A(e_i)\iota_X(\xi)) = \iota_X(\xi) = \lim(\iota_X(\xi)\iota_A(e_i)).$$

So we have surjectivity as required. \square

The remainder of this section is dedicated to classifying extensions of Hilbert bimodules. This is done in [2] for right Hilbert modules, and as with the rest of this section we simply show that the same constructions and methods work for Hilbert bimodules.

Proposition 3.4. *Given an extension (Y, B) of a full Hilbert bimodule (X, A) , there is a unique morphism of Hilbert bimodules*

$$(\lambda_Y, \lambda_B) : (Y, B) \rightarrow (M(X), M(A))$$

such that $(\lambda_Y, \lambda_B) \circ (\psi_X, \psi_A) = (\iota_X, \iota_A)$. Furthermore, (λ_Y, λ_B) is injective if and only if (Y, B) is essential.

Proof. We know from [3, Theorem 1.1] that there is a morphism of right Hilbert modules

$$(\lambda_Y, \lambda_B) : (Y, B) \rightarrow (M(X), M(A))$$

satisfying

$$\lambda_Y(\xi)(a) = \psi_X^{-1}(\xi\psi_A(a))\lambda_B(b)(a) = \psi_A^{-1}(b\psi_A(a)).$$

It is shown in [3] that $\lambda_Y(\xi)$ is adjointable $\lambda_Y(\xi)^* \in \mathcal{L}(X, A)$ satisfying

$$\lambda_Y(\xi)^*(\alpha) = \psi_A^{-1}(\langle \xi, \psi_X(\alpha) \rangle_Y).$$

We need only check that these maps also preserve the left module structure. For the left inner product, first notice that since we assume that (X, A) is full, we have an isomorphism $\pi : A \rightarrow \mathcal{K}(X)$, so it is enough to show that for any $\alpha, \beta \in X$ and $\xi, \eta \in Y$ we have

$$\pi(\lambda_B({}_Y\langle \xi, \eta \rangle)_X\langle \alpha, \beta \rangle) = \lambda_Y(\xi)\lambda_Y(\eta)^*\theta_{\alpha, \beta}.$$

Calculating, we see

$$\begin{aligned} \lambda_B({}_Y\langle \xi, \eta \rangle)_X\langle \alpha, \beta \rangle &= \psi_X^{-1}({}_B\langle \xi, \eta \rangle\psi_X({}_X\langle \alpha, \beta \rangle)) \\ &= \psi_X^{-1}({}_Y\langle \xi\langle \eta, \psi_X(\alpha) \rangle_Y, \psi_X(\beta) \rangle) \\ &= \psi_X^{-1}({}_Y\langle \xi\psi_A(\lambda_Y(\eta)^*(\alpha)), \psi_X(\beta) \rangle) \\ &= {}_X\langle \lambda_Y(\xi)\lambda_Y(\eta)^*(\alpha), \beta \rangle \end{aligned}$$

Applying the isomorphism π to both sides we get

$$\pi(\lambda_B({}_Y\langle \xi, \eta \rangle)_X\langle \alpha, \beta \rangle) = \theta_{\lambda_Y(\xi)\lambda_Y(\eta)^*(\alpha), \beta} = \lambda_Y(\xi)\lambda_Y(\eta)^*\theta_{\alpha, \beta}$$

as required. For the left action, fix some $\xi \in Y, b \in B$ and $a \in A$. Then

$$\begin{aligned} \lambda_Y(b\xi)(a) &= \psi_X^{-1}((b\xi)\psi_A(a)) \\ &= \psi_X^{-1}(b(\xi)\psi_A(a)) \\ &= \lambda_B(b)\psi_X^{-1}(\xi\psi_A(a)) \\ &= \lambda_B(b)\lambda_Y(\xi)(a) \end{aligned}$$

as required. So the proof is finished. \square

Proposition 3.5. *Let (Y, B) be an extension of a Hilbert bimodule (X, A) . Then with the structure inherited from (Y, B) , the pair of quotient spaces $(Y/X, B/A)$ is also a Hilbert bimodule. Furthermore, the pair of quotient maps $(q_Y, q_B) : (Y, B) \rightarrow (Y/X, B/A)$ is a morphism in the category of Hilbert bimodules.*

Proof. We begin by defining the left and right actions, and the left and right inner-products. For $q_Y(\xi), q_Y(\eta)$ in Y/X , define

$$\begin{aligned} {}_{Y/X}\langle q_Y(\xi), q_Y(\eta) \rangle &= q_B({}_Y\langle \xi, \eta \rangle) \\ \langle q_Y(\xi), q_Y(\eta) \rangle_{Y/X} &= q_B(\langle \xi, \eta \rangle_Y). \end{aligned}$$

Likewise, for $q_B(b) \in B/A$, define

$$\begin{aligned} q_B(b)q_Y(\xi) &= q_Y(b\xi) \\ q_Y(\xi)q_B(b) &= q_Y(\xi b). \end{aligned}$$

Notice that these actions are well-defined since we have assumed that $\psi_A(A)$ is Y -invariant, and the non-degeneracy of the left and right actions on X implies that $B\psi_X(X) = \psi_X(X)B = \psi_X(X)$. The well-definedness of the inner-products is clear from the definition.

Secondly, given $q_Y(\xi), q_Y(\eta)$ and $q_Y(\zeta)$ in Y/X , we have

$$\begin{aligned} q_Y(\xi)\langle q_Y(\eta), q_Y(\zeta) \rangle_{Y/X} &= q_Y(\xi\langle \eta, \zeta \rangle_Y) \\ &= q_Y({}_Y\langle \xi, \eta \rangle \zeta) \\ &= {}_{Y/X}\langle q_Y(\xi), q_Y(\eta) \rangle q_Y(\zeta) \end{aligned}$$

as required. So we have a Hilbert bimodule. \square

This construction motivates the following definition.

Definition 3.6. For a Hilbert bimodule (X, A) , we denote the quotient spaces by $Q(X) := M(X)/X$ and $Q(A) := M(A)/A$. We call the Hilbert bimodule $(Q(X), Q(A))$ the *corona bimodule*. When (X, A) is full, for an extension (Y, B) of (X, A) there is a unique morphism $(\delta_Y, \delta_B) : (Y/X, B/A) \rightarrow (Q(X), Q(A))$ satisfying

$$(\delta_Y, \delta_B) \circ (q_Y, q_B) = (q_{M(X)}, q_{M(A)}) \circ (\lambda_Y, \lambda_B).$$

We call (δ_Y, δ_B) the *Busby morphism* associated to the extension (Y, B) . It is straightforward to see that this is indeed a morphism.

Now, suppose we have Hilbert bimodules $(X, A), (Y, B)$ and (Z, C) , and morphisms

$$(\psi_X, \psi_A) : (X, A) \rightarrow (Z, C) \text{ and } (\omega_Y, \omega_B) : (Y, B) \rightarrow (Z, C).$$

Then we can form the *restricted direct-sum*

$$X \oplus_Z Y := \{(\xi, \eta) \in X \oplus Y : \psi_X(\xi) = \omega_Y(\eta)\}.$$

If we also form the pullback C^* -algebra

$$A \oplus_C B = \{(a, b) \in A \oplus B : \psi_A(a) = \omega_B(b)\}$$

then the pair $(X \oplus_Z Y, A \oplus_C B)$ with left and right inner-products

$$(1) \quad {}_{X \oplus_Z Y} \langle (\alpha, \beta), (\xi, \eta) \rangle = ({}_X \langle \alpha, \xi \rangle, {}_Y \langle \beta, \eta \rangle) \langle (\alpha, \beta), (\xi, \eta) \rangle_{X \oplus_Z Y} = (\langle \alpha, \xi \rangle_X, \langle \beta, \eta \rangle_Y)$$

and left and right actions

$$(a, b)(\xi, \eta) = (a\xi, b\eta)(\xi, \eta)(a, b) = (\xi a, \eta b)$$

is a Hilbert bimodule. For a proof of this, see [13, Proposition 3.2] Furthermore, it is easy to check that the maps

$$(p_X, p_A) : (X, A) \rightarrow (X \oplus_Z Y, A \oplus_C B)$$

and

$$(p_Y, p_B) : (Y, B) \rightarrow (X \oplus_Z Y, A \oplus_C B)$$

which are just the projections onto the first and second coordinates respectively, are morphisms of Hilbert bimodules. Note that these maps are not surjective in general.

Proposition 3.7. *Let (X, A) be a full Hilbert bimodule, (Z, C) be a Hilbert bimodule, and $(\delta_Z, \delta_C) : (Z, C) \rightarrow (Q(X), Q(A))$ be a morphism of Hilbert bimodules. Then there exists an extension (Y, B) of (X, A) whose Busby morphism is (δ_Z, δ_C) . Furthermore, this extension is essential if and only if (δ_Z, δ_C) is injective.*

Proof. This result is just the Hilbert bimodule version of [2, Proposition 3.4], and the proof is more or less the same. Namely,

$$(Y, B) := (M(X) \oplus_{Q(X)} Z, M(A) \oplus_{Q(A)} C)$$

satisfies the required properties. The second claim follows directly from [2, Proposition 3.3]. \square

4. C^* -ALGEBRAS ASSOCIATED TO HILBERT BIMODULES

We begin with the definition of a covariant representation of a Hilbert bimodule.

Definition 4.1. A *covariant representation* of a Hilbert bimodule (X, A) on a C^* -algebra D is a morphism $(t_X, t_A) : (X, A) \rightarrow (D, D)$, where the pair (D, D) has the structure of a Hilbert bimodule as described in Example 2.2.

Definition 4.2. Let (X, A) be a Hilbert-bimodule. The Cuntz-Pimsner algebra \mathcal{O}_X associated to (X, A) is defined to be the universal C^* -algebra generated by covariant representations. We denote the universal covariant representation of (X, A) by

$$(T_X, T_A) : (X, A) \rightarrow (\mathcal{O}_X, \mathcal{O}_X).$$

For the proof of the existence of such a universal algebra, see [8].

Proposition 4.3. *Given a morphism $(\psi_X, \psi_A) : (X, A) \rightarrow (Y, B)$ there exists a unique C^* -homomorphism $\Psi : \mathcal{O}_X \rightarrow \mathcal{O}_Y$ such that the following diagram is commutative.*

$$\begin{array}{ccc}
 (X, A) & \xrightarrow{(\psi_X, \psi_A)} & (Y, B) \\
 \downarrow (T_X, T_A) & & \downarrow (T_Y, T_B) \\
 \mathcal{O}_X & \xrightarrow{\Psi} & \mathcal{O}_Y
 \end{array}$$

The proof of this proposition can be found in [13, Proposition 2.9].

In what follows, we will denote morphisms between Hilbert bimodules as lower-case characters, and the corresponding homomorphism between Cuntz-Pimsner algebras will be denoted by the corresponding upper-case character.

Theorem 4.4. *Let $(\psi_X, \psi_A) : (X, A) \rightarrow (Y, B)$ be a morphism of Hilbert-bimodules. Then $(\ker(\psi_X), \ker(\psi_A))$ with left and right actions, and left and right inner products inherited from (X, A) is also a Hilbert bimodule. Furthermore, $\mathcal{O}_{\ker(\psi_X)} \cong \ker(\Psi)$.*

Proof. We begin by recalling that the C^* -algebra \mathcal{O}_X admits a gauge action; that is, there exists a map

$$\alpha : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_X)$$

such that

$$\alpha_z(T_X(\xi)) = zT_X(\xi) \quad \text{and} \quad \alpha_z(T_A(a)) = T_A(a)$$

for any $z \in \mathbb{T}, \xi \in X$ and $a \in A$. Hence, by [8, Proposition 10.6], it is enough to show that the ideal $\ker(\Psi)$ is invariant under the gauge action. This follows easily by noticing that for any $z \in \mathbb{T}, \alpha_z$ commutes with Ψ . \square

It is worth noting that the previous theorem is not true in general for arbitrary C^* -correspondences, and it is for this reason that this study is restricted to the class of Hilbert bimodules.

We easily get the following corollary.

Corollary 4.5. *Let (Y, B) be an extension of a Hilbert bimodule (X, A) . Then the short exact sequence of Hilbert bimodules*

$$0 \longrightarrow (X, A) \xrightarrow{(\psi_X, \psi_A)} (Y, B) \xrightarrow{(q_Y, q_B)} (Y/X, B/A) \longrightarrow 0$$

induces a short exact sequence of Cuntz-Pimsner algebras

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\Psi} \mathcal{O}_Y \xrightarrow{Q} \mathcal{O}_{Y/X} \longrightarrow 0$$

Consequently, we have an isomorphism $\mathcal{O}_{Y/X} \cong \mathcal{O}_Y/\mathcal{O}_X$.

Example 4.6. Let A be a C^* -algebra and $\alpha \in \text{Aut}(A)$ be an automorphism. Let $X = A$. Then (X, A) has left and right inner-products and right action as in example 2.2. The automorphism α define a left action $x \cdot a = \alpha(x)a$, and we have a Hilbert bimodule. It is well-know (see [11] for example) that \mathcal{O}_X is isomorphic to the crossed-product algebra $A \rtimes_\alpha \mathbb{Z}$.

Now suppose we have some ideal $I \triangleleft A$ with $\alpha(I) = I$ and $AI = IA$. With $W = I$ and considering (W, I) as a Hilbert bimodule as above, it is easily seen that (X, A) is an extension of (W, I) . Since $\alpha(I) = I$, α also defines an automorphism of A/I . So Corollary 4.5 tells us that the well-known exact sequence of C^* -algebras

$$0 \longrightarrow I \rtimes_\alpha \mathbb{Z} \longrightarrow A \rtimes_\alpha \mathbb{Z} \longrightarrow A/I \rtimes_\alpha \mathbb{Z} \longrightarrow 0$$

comes from the exact sequence of Hilbert bimodules

$$0 \longrightarrow (W, I) \longrightarrow (X, A) \longrightarrow (X/W, A/I) \longrightarrow 0.$$

We end this section with some results regarding the Cuntz-Pimsner algebra $\mathcal{O}_{M(X)}$ associated to the multiplier bimodule $(M(X), M(A))$.

Proposition 4.7. *Let (X, A) be a full Hilbert bimodule. Then there exists an injective covariant representation $(\overline{T_X}, \overline{T_A})$ of $(M(X), M(A))$ on the multiplier algebra $M(\mathcal{O}_X)$. Furthermore, $\mathcal{O}_{M(X)} \cong C^*(\overline{T_X}, \overline{T_A})$.*

Proof. We begin by showing that $T_A : A \rightarrow \mathcal{O}_X$ is non-degenerate; that is $T_A(A)\mathcal{O}_X = \mathcal{O}_X = \mathcal{O}_X T_A(A)$. Given that \mathcal{O}_X is generated by the image of X it is enough to show that $T_A(A)T_X(X) = T_X(X) = T_X(X)T_A(A)$. This easily follows from the non-degeneracy of the left and right actions.

Hence, if we think of the pair (T_X, T_A) as a morphism $(X, A) \rightarrow (M(\mathcal{O}_X), M(\mathcal{O}_X))$, [6, Theorem 1.30] implies that there is an extended morphism $(\overline{T_X}, \overline{T_A}) : (M(X), M(A)) \rightarrow (M(\mathcal{O}_X), M(\mathcal{O}_X))$; i.e. a representation of $(M(X), M(A))$ on $M(\mathcal{O}_X)$.

It remains to see that this representation is injective. Notice that $\ker(\overline{T_A}) \triangleleft M(A)$ satisfies $\ker(\overline{T_A}) \cap A = \{0\}$, and since A is an essential ideal in $M(A)$, we must have $\ker(\overline{T_A}) = \{0\}$. So the representation is injective.

Finally, given the gauge-invariant uniqueness theorem [10, Theorem 6.4], to see that we have an isomorphism $\mathcal{O}_{M(X)} \cong C^*(\overline{T_X}, \overline{T_A})$, it is enough to show that the injective representation $(\overline{T_X}, \overline{T_A})$ admits a gauge action. Let $\gamma : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_X)$ denote the gauge action on \mathcal{O}_X . Then for fixed $z \in \mathbb{T}$, it is easily shown that the pair $(\gamma_z \circ T_X, \gamma_z \circ T_A)$ is an injective representation of (X, A) on \mathcal{O}_X . So it extends to an

injective representation $(\overline{\gamma_z \circ T_X}, \overline{\gamma_z \circ T_A})$ of $(M(X), M(A))$. It is easily checked that this representation satisfies

$$\overline{\gamma_z \circ T_X}(T) = zT \text{ and } \overline{\gamma_z \circ T_A}(m) = m$$

for all $z \in \mathbb{T}, T \in M(X)$ and $m \in M(A)$. This shows that the representation admits a gauge action as required. \square

As a corollary to this result we have the following.

Corollary 4.8. *Let (X, A) be a full Hilbert bimodule and (Y, B) be an extension of (X, A) . Then we have an isomorphism*

$$\mathcal{O}_Y \cong \mathcal{O}_{M(X)} \oplus_{\mathcal{O}_{Q(X)}} \mathcal{O}_{Y/X}$$

where the pull-back is taken along $\Lambda : \mathcal{O}_Y \rightarrow \mathcal{O}_{M(X)}$ and the induced Busby map $\Delta : \mathcal{O}_{Y/X} \rightarrow \mathcal{O}_{Q(X)}$.

Proof. It is well-known that there is an isomorphism

$$\mathcal{O}_Y \cong M(\mathcal{O}_X) \oplus_{\mathcal{O}_{Q(X)}} \mathcal{O}_Y/\mathcal{O}_X.$$

The previous proposition implies we have $\mathcal{O}_{M(X)} \rightarrow M(\mathcal{O}_X)$ injective, and Corollary 4.5 says we have an isomorphism $\mathcal{O}_Y/\mathcal{O}_X \cong \mathcal{O}_{Y/X}$. Hence it is enough to show that the image of the Busby map Δ has image inside $\mathcal{O}_{Q(X)}$ and this is easily checked. \square

In light of Proposition 4.7, one may ask whether the injective map $\mathcal{O}_{M(X)} \rightarrow M(\mathcal{O}_X)$ extends to an isomorphism. We end the paper with an example illustrating that this is not the case in general.

Example 4.9. Let $A = X = C_0(\mathbb{N})$. For $f \in A$ and $\xi, \eta \in X$ define the left and right actions by

$$(f\xi)(n) = f(n)\xi(n) \text{ and } (\xi f)(n) = \xi(n)f(n+1)$$

and left and right inner-products

$${}_X\langle \xi, \eta \rangle(n) = \xi(n)\overline{\eta(n)} \text{ and } \langle \xi, \eta \rangle_X(n) = \overline{\xi(n-1)}\eta(n-1).$$

For $n \in \mathbb{N}$, let ξ_n, f_n be the characteristic functions in X and A respectively. Define elements of \mathcal{O}_X

$$\begin{aligned} E_{ii} &= T_A(f_i) \\ E_{ij} &= T_X(\xi_i) \dots T_X(\xi_{j-1}) \text{ for } i < j \\ E_{ij} &= T_X(\xi_{i-1})^* \dots T_X(\xi_j)^* \text{ for } i > j. \end{aligned}$$

Then $\{E_{ij} : i, j \in \mathbb{N}\}$ are a system of matrix units generating \mathcal{O}_X , so we get an isomorphism $\mathcal{O}_X \cong \mathcal{K}(\ell^2(\mathbb{N}))$. Let $\{e_i : i \in \mathbb{N}\}$ be an orthonormal basis for $\ell^2(\mathbb{N})$.

Now let $S \in M(X) = \mathcal{L}(A, X)$. Then since S is right A -linear, we must have

$$S(f_n) = \lambda_n \xi_{n-1}$$

for some $\lambda_n \in \mathbb{C}$. So we see that $M(X) = C_b(\mathbb{N})$ - bounded functions on the natural numbers. We also have $M(A) = C_b(\mathbb{N})$. Then we have

the extended representation $(\overline{T_X}, \overline{T_A})$ of the bimodule $(M(X), M(A))$ on $M(\mathcal{K}(\ell^2(\mathbb{N}))) = \mathcal{B}(\ell^2(\mathbb{N}))$. If $S \in M(X) = C_b(\mathbb{N})$,

$$\overline{T_X}(S)(e_i) = S(i)e_{i+1}.$$

is a so-called weighted-shift operator. So Proposition 4.7 implies that $\mathcal{O}_{M(X)}$ is isomorphic to the C^* -subalgebra of $\mathcal{B}(\ell^2(\mathbb{N}))$ generated by the weighted-shift operators. It is shown in [5] that this is strictly smaller than $\mathcal{B}(\ell^2(\mathbb{N}))$. So in this case, $\mathcal{O}_{M(X)} \not\cong M(\mathcal{O}_X)$.

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